Exploded Points

Okan Gurel
IBM Cambridge Scientific Center, Cambridge, Mass. U.S.A.

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A new concept of a mathematical object of zero dimension, an exploded point, is introduced. The dimension used is defined on the basis of the functional characteristics of the system, thus it may be referred to as f-dimension. A stability index is also defined for the mathematical objects including exploded points, which can be related to the f-dimension. It is shown that the mathematical object exhibited by the Lorenz system after the second bifurcation is such a point. A recursive formula based on the definition of the exploded point.

In this note the concept of an exploded point, a mathematical object, is introduced. These objects have been observed in some dynamical systems exhibiting oscillations without any specific period. Some mathematical preliminaries necessary for the definition of an exploded point are given below.

Let us assume that a dynamical system is given in E^n, as characterized by $f_p\colon X \to X$, $X \subset E^n$. For some part of the parameter space P there exists only one singular point x_0, asymptotically stable or stable in the sense of Liapunov.

Definition 1a: A *singular point* x_0 is a solution to the equation $f_p = 0$, thus $f_p\colon x_0 \to x_0$.

Linearized system, Lf_p at x_0 is used in obtaining the characteristic equation $|Lf_p - \lambda|_{x_0} = 0$. The n eigenvalues are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. Referring to the characteristic orientations, and using the notation in Reference [1 a], for each eigenvalue there exists a pair of 1/2 flows denoted by $1/2\,w_s$ or $1/2\,w_u$, approaching or leaving x_0, respectively. i.e. stable and unstable 1/2 flows. Each pair forms either stable or unstable flows denoted by $w^1_{m_1}$ or $w^1_{m_2}$, respectively.

Definition 1b: The singular point x_0 is embedded into the n-dimensional solution space X, $x_0 \in X \subset E^n$ with n (stable or unstable) flows $w^1_{m_1}$ or $w^1_{m_2}$, $(n-k)\,w^1_{m_1} + k\,w^1_{m_2}$, $k = 0$ implying stable singular solution.

To start with all the eigenvalues, real or imaginary, of the linearized system in the neighborhood of x_0 are assumed to lie in the negative half plane. Thus $k = 0$, and x_0 is characterized by $n \times w^1_{m_1}$. Further we assume that x_0 is a generating singular point in that for some boundaries in P, it bifurcates to yield certain bifurcated solutions.

Definition 2: An *exploded point* x_e is a solution as $f_p\colon x_e \to x_e$, and it is embedded into the n-dimensional solution space $x_e \in X \subset E^n$, with *only one* (stable or unstable) flow $w^1_{m_1}$ or $w^1_{m_2}$, approaching or leaving it, respectively.

Definition 3: A *limit cycle*, L is a solution as $f_p\colon L \to L$, and it is embedded into the n-dimensional solution space $L \in X \subset E^n$, with *only two* (stable or unstable) flows $w^1_{m_1}$ or $w^1_{m_2}$, approaching or leaving it, respectively.

Definition 4: The f-dimension (functional dimension) of solution x is one less than the combined dimension of the stable and unstable manifolds d, which is the number of approaching and leaving flows, thus $f\text{-dim}\,x = d - 1$.

Remark 1. $f\text{-dim}\,x_0 = n - 1$ a singular point (notice this is the dimension of the boundary of X);

$f\text{-dim}\,x_e = 1 - 1 = 0$, an exploded point;
$f\text{-dim}\,L = 2 - 1 = 1$, a limit cycle (a closed line).

Remark 2. An exploded point does not satisfy $f_p = 0$, thus it is not stationary but oscillating, however without any period.

An Index, an Associated Object and Exploded Points

The Poincaré index [2] is an easily calculable characteristic of a singular point. Here we define an index which incorporates also the stability of the singular points.

Reprint requests to Prof. Dr. O. Gurel, IBM Cambridge Scientific Center, 545 Technology Square, Cambridge, Mass. 02139 U.S.A.

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Definition 5a (A Stability Index): *Stability index of a singular point* is defined as the difference between the number of $w^1_{m_1}$, i.e. the dimension of the stable manifold, $(n-k)$, and that of $w^1_{m_2}$, i.e. the dimension of the unstable manifold, k,

$$I_s = (n-k) - k = n - 2k\,.$$

Remark 1. Stability index for various singular points are:

$k = 0$ *stable focus* of the stability index n, $I_s = n$,

$k = n$ *unstable focus* of the stability index n, $I_s = -n$,

$0 < k < n$ a *saddle* with the stability index $n - 2k$.

Remark 2. The only stable singular point is when $I_s = n$.

Definition 5b: Stability index of a set of m multiple singular points is defined as the sum of indices of the individual singular points,

$$I_s{}^m = \sum_{i=1}^{m} (n_i - 2k_i)\,.$$

Definition 5c: A *dual index* $I_s{}^*$ is the index whose sum with an index I_s is equal to zero,

$$I_s + I_s{}^* = 0\,.$$

Definition 6: An *associated object* with another object whose index is I_s is defined as that with a dual index $I_s{}^*$.

Proposition. If bifurcation (peeling) of a singular point (or cooperative peeling of multiple singular points) does not result in additional singular points, the newly created object (or a set of objects) forms an associated object, i.e. the index of the associated object is equal to $I_s{}^*$.

Definition 7: A stable (or unstable) exploded point x_e is an associated object with a stability index $I_s = +1$ (or -1).

We also give a theorem stating the necessary and sufficient conditions for the existence of an exploded points.

Theorem. A system S, stable in a finite domain $\Omega \subset E^n$, having only one singular point (or m multiple singular points) with k positive, $(n-k)$ negative eigenvalues (or cooperatively $K = k_1 + \cdots + k_m$ positive, $N = mn - K$ negative eigenvalues) possesses an exploded point in Ω if and only if $k = 1$ (or $K - N = 1$).

Proof is trivial. If the system S is stable, there is an attracting region Ω where all flows must enter and remain thereafter. If there is only one positive eigenvalue $w^1_{m_2}$ of a singular point (or collectively from all the singular points) leaving the singular point (or multiple singular points collectively), it can not go to the outside of Ω and by definition it remains in an object whose dimension is 0, thus an exploded point is reached.

A remark is proper at this point. It is quite possible that at one point there might be two flows leaving a singular point. However rather than reaching a limit cycle (see Definition 3) there might be two exploded points. This is possible only if the two exploded points appear during a series of bifurcations rather than simultaneously during a single bifurcation. Thus the theorem is still valid.

Now we consider various possibilities of stability changes associated with the bifurcation of the generating singular point. The following cases are of interest:

1. $n \times w^1_{m_1} \to n \times w^1_{m_2}$, i.e. all the eigenvalues cross the imaginary axis, thus the point is now completely unstable with an n-dimensional unstable manifold. When accompanied by a change in the number of critical solutions this may lead to peeling [1b].

2. $n \times w^1_{m_1} \to k \times w^1_{m_2} \oplus (n-k) \times w^1_{m_1}$, i.e. only k eigenvalues cross the imaginary axis, thus a k-dimensional unstable manifold and an $(n-k)$-dimensional stable manifold exist. With an alteration in the number of critical solutions this may lead to partial peeling [1c].

It should be noted that each $w^1_{m_2}$ leaving the unstable point approaches a point on the stable limiting set. Thus n such points form the simplex of the limiting set whose *f*-dimension is therefore $(n-1)$.

In the first case, let us assume that there is no other singular point bifurcating, however only one limiting set forms. By the global stability theorem [1b], the stability property of this new objects is same as that of the generating singular point, thus it is a stable hypersurface with the *f*-dimension $(n-1)$. If for example, $n = 3$, the stable object is a 2-dimensional surface, if $n = 2$, 1-dimensional closed curve, a stable limit cycle.

In the second case, it is again assumed that no other singular point bifurcates, and around the bifurcating singular point only one stable limiting set forms. We therefore observe that if $k = 1$ there can be 0-dimensional objects with a proper stability property that they can represent exploded points. An example is the Lorenz attractor. Following the reasoning above we can show that the generating singular point of the Lorenz model goes through bifurcations as follows:

$r < 1$: $3 \times w^1_{m_1}$ (generating singular point),

$r > 1$: $2 \times w^1_{m_1} \oplus 1 \times w^1_{m_2}$ (generating singular point) and

$2 \times [3 \times w^1_{m_1}]$ (two additional singular points are created).

These three singular points cooperatively have

$$(2 + 6) \times w^1_{m_1} \oplus 1 \times w^1_{m_2} = 7 \times w^1_{m_1}$$

which is a completely stable property. The next bifurcation is the bifurcation (peeling) of these three singular points "cooperatively", [1d].

$r > r_c$: $2 \times w^1_{m_1} \oplus 1 \times w^1_{m_2}$ (original generating singular point),

$2 \times [1 \times w^1_{m_1} \oplus 2 \times w^1_{m_2}]$ (two additional singular points).

Cooperatively $(2 + 2) \times w^1_{m_1} \oplus (1 + 4) \times w^1_{m_2}$ yield $1 \times w^1_{m_2}$. Since this is an example of bifurcation, this unstable manifold must be attracted by an additional object, the fourth critical solution lying in a finite domain. The Lorenz attractor thus has its f-dimension as 0, the dimension of the manifold minus 1. Thus the Lorenz attractor is actually a point however, not singular but exploded.

The topology of an exploded point is not clear. In fact, what we can say is that neither periodicity nor almost periodicity should be expected, which seems to be the case with the Lorenz attractor. In addition to the question of topology of the inside of an exploded point there is a quantitative question. How many different exploded points should be expected for a given problem? This depends on the pairwise combination of stable and unstable characteristic orientations of the solutions making up the limiting set. For example, for the Lorenz system, taking stable und unstable characteristic orientations for S_0, S_1, S_2 and the Lorenz attractor as on Table 1, and indicating $u \to s$ by an entry

Table 1.

		S_0 s	s	S_1 s	S_2 s	LA s
S_0	u	1	1	①	①	1
S_1	u	1	①	①	1	1
	u	①	1	1	1	①
S_2	u	1	①	1	①	1
	u	①	1	1	1	①

as 1, we see that a number of combinations may be obtained in covering the matrix completely, such that each row and each column adds up to 2. The circled configuration is represented in Figure 1. A recurrence formula for all the possible configurations can be deductively obtained as

$$N_n = (K_n/2)\, n!$$

where n is the number of columns (and rows) and the relationship for the coefficient K_n is

$$K_n = K_{n-2} + 3 \quad \text{for} \quad n \geqq 5, \quad K_1 = 0,$$
$$K_2 = 1, \quad K_3 = 2, \quad K_4 = 3.$$

This gives the result as

$$n = 2, \quad n = 4, \quad n = 5, \quad n = 6 \ldots$$
$$N_2 = 1, \quad N_4 = 36, \quad N_5 = 300, \quad N_6 = 2160 \ldots$$

Therefore the Lorenz system is one of the 300 possible configurations! The sketch on Fig. 1 is a representative scheme corresponding to the numerical results obtained for the Lorenz model, see e.g. [1e].

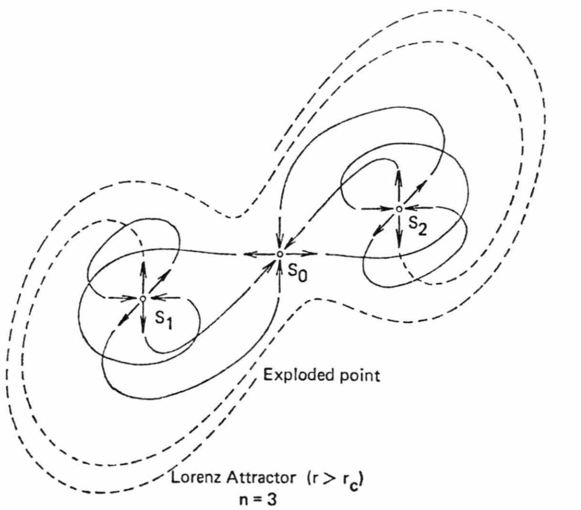

Fig. 1

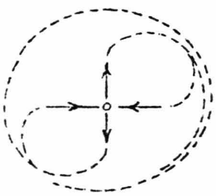

Fig. 2

Another example is the homoclinic limit cycle [1b]. As it can be seen this limit cycle is the combination of the two exploded points, one with $w^1_{m_1}$ the other with $w^1_{m_2}$, obtained following the bifurcation of a saddle point. It consists of infinitely many points in the finite domain without really possessing any periodicity in the classical sense, and each point has both stable and unstable manifolds. It is a pathological example, however as important as the other attractors such as the Lorenz attractor.

The "chaotic" behavior of these attractor may well be due to the explosion of a point, however "chaos" as discussed in the literature may include some other possibilities.

The example in [1c] should also be considered that following a partial peeling,

$$6 \times w^1_{m_1} \to 4 \times w^1_{m_1} \oplus 2 \times w^1_{m_2}.$$

The unstable manifold is attracted in the finite domain by a 1-dimensional stable object, the limit cycle, as expected. Therefore it is not an example of exploded points.

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[2] V. V. Nemytskii and V. V. Stepanov, Qualitative Theory of Differential Equations, Princeton University Press, 1960.